of the normal force is considered. The relative damping of the two possible coning motions depends on the relative size of $C_{N\alpha}$ and C_{m_q} and either coning motion can display the instability discussed by Platus.

In Ref. 6 Phillips briefly considered the dynamic stability of spinning shell and makes the erroneous remark that all shells are dynamically unstable. In this case only the damping moment was considered. If the proper damping effect of the normal force had been inserted, he would have shown the correct result that most shells are dynamically stable. Thus damping induced by the normal force has a number of important consequences.

In both cases considered, a complete linear analysis should include drag and a linear Magnus moment coefficient. The Magnus coefficient combines with the additional normal force term in the yaw moment and can induce dynamic instability. Nonlinear Magnus moments can cause even more interesting behavior such as limit cycles and are discussed in detail in the literature.

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Reply by Author to C. H. Murphy

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MURPHY points out that a small additional term should be included to describe completely normal force damping of the yaw motion. He then concludes that omission of this term is the cause of the precession instability discussed in Ref. 1. This instability is caused by the fact that negative precession motion is damped less than positive precession motion, as Murphy states above, but a second condition is also required for the instability to occur. The oscillation amplitude of the precession rate $\dot{\psi}$ must increase and/or the reduced roll rate p_r must increase such that the minimum value of the oscillation $y = \dot{\psi} - p_r$ approaches zero. The relative damping of the two precession motions depends on the roll acceleration $\dot{p}$ in addition to the relative size of $C_{N\alpha}$ and C_{m_q} (and Magnus effects, if present). Positive roll acceleration contributes to increased damping of positive precession and decreased damping of negative precession and, at the same time, causes the roll rate parameter p_r to increase and reach the lower envelope of the $\dot{\psi}$ oscillation, thereby inducing the instability discussed in Ref. 1. This instability can still occur if the complete effects of normal force damping are included, as shown below. It can also occur in the opposite direction, as Murphy points out. However, the importance of the damping induced by roll acceleration should be recognized. The quantitative relation between the various damping effects on the precession motion, including the complete effects of normal force damping and a linear Magnus moment coefficient, is derived here as an extension to Ref. 1.

The yaw moment equation, including the complete effects of normal force damping and a linear Magnus moment coefficient $C_{n_{p\alpha}}$, can be written

$$M_\zeta = -(qSd^2/2u)[(-C_{m_q} + C_{N\alpha}')\dot{\psi}\theta - (\mu C_{N\alpha}' + C_{n_{p\alpha}})p\theta] \quad (1)$$

where $\mu \equiv I_x/I$ and $C_{N\alpha}' \equiv 2C_{N\alpha}I/md^2$. The small angle-of-attack pitch and yaw equations of motion can then be written

$$\begin{aligned} \ddot{\theta} + (\Omega^2 - y^2)\theta + \nu\dot{\theta} &= 0 \\ (\dot{y} + \nu y + \xi)\theta + 2y\dot{\theta} &= 0 \end{aligned} \quad (2)$$

where

$$\begin{aligned} \Omega \equiv (\omega^2 + p_r^2)^{1/2},\ y &\equiv \dot{\psi} - p_r,\ \xi \equiv \dot{p}_r + \nu p_r - \nu_m p \\ \nu &\equiv (qSd^2/2Iu)(-C_{m_q} + C_{N\alpha}') \\ \nu_m &\equiv (qSd^2/2Iu)(\mu C_{N\alpha}' + C_{n_{p\alpha}}) \end{aligned} \quad (3)$$

Equations (2) can be combined by eliminating θ to give the single equation in y and $v = \dot{y}$

$$dv/dy = [-4y^2(y^2 - \Omega^2) - (\nu^2y^2 - \xi^2) - 2y\xi + v(3v + 4\xi)]/(2yv) \quad (4)$$

suitable for phase plane analysis.

Stability of the two precession motions is determined by the character of the singularities at $v = 0$ and $y = \pm(\Omega - \epsilon)$, where $\epsilon \approx (\nu^2\Omega^2 \pm 2\xi\Omega - \xi^2)/(8\Omega^3)$ is small, in general, compared with Ω. Both singularities are of the spiral or center types with stability determined by the sign of the parameter ξ. For $\xi = 0$ both singularities are centers, indicating stable oscillations. For $\xi > 0$ the singularity at $y = \Omega - \epsilon$ is an unstable spiral and that at $y = -\Omega + \epsilon$ is a stable spiral, and for $\xi < 0$ the reverse is true. From the definition of ξ, Eq. (3), the stability criterion can be written

$$\begin{aligned} &\xi = p_r[(\dot{p}/p) + \nu - 2(\nu_m/\mu)] \\ &\xi = 0 \text{ both modes stable} \\ &\xi > 0 \begin{cases} \text{positive mode unstable} \\ \text{negative mode stable} \end{cases} \\ &\xi < 0 \begin{cases} \text{negative mode unstable} \\ \text{positive mode stable} \end{cases} \end{aligned} \quad (5)$$

This indicates that positive precessional motion grows and negative precessional motion decays for $\xi > 0$ and vice-versa for $\xi < 0$.† The only other singularities in the phase plane are an unstable node at $v = -\xi$ and an unstable saddle at $v = -\xi/3$ along the v axis. Consequently, an oscillation in $\dot{\psi}$ in either the positive or negative mode will persist until $y = \dot{\psi} - p_r$ reaches the limiting value $y = 0$, which is a necessary condition for the instability to occur. The precession instability so defined is a distinct reversal from the "positive precession mode" to the "negative precession mode," or vice-versa, and is not considered here to be merely a growth of one precession arm and a decay of the other. The fact that

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† A divergence of the $\dot{\psi}$ oscillation in the positive precession mode ($\xi > 0$) actually corresponds to a growth of the (negative) precession arm and a decay of the nutation arm in Murphy's notation; i.e., the transition is from the less stable to the more stable mode.

the motion persists in either a positive or negative precession mode is apparent from the phase plane behavior as characterized by the two spiral-type singularities. The motion in either mode, in general, consists of a superposition of the two coning motions, as described by Murphy's Eq. (7) above. Nevertheless, the distinction is made that the motion is either "positive precession" or "negative precession," depending on whether y is, in general, positive or negative, respectively.

The damping term ν_m was not included in the analysis of Ref. 1. In the absence of Magnus effects, $\nu_m p = 2f\nu p_r$, where $f \equiv C_{N\alpha}'/(C_{N\alpha}' + C_{mq})$, and the stability criterion can be written

$$\xi = p_r[\dot{p}/p + \nu(1 - 2f)] \tag{6}$$

Therefore, for $C_{N\alpha}' \leq C_{mq}$ or $f \leq \frac{1}{2}$, ξ is always positive for positive $\dot{p}/p$ and the instability always occurs from the positive to the negative mode. With $C_{N\alpha}' > C_{mq}$ or with the presence of Magnus effects, the instability could occur in either direction, as Murphy points out. However, for sufficiently large $\dot{p}/p$, ξ can always become positive regardless of the magnitudes of C_{mq}, $C_{N\alpha}$, and $C_{np\alpha}$. In either case, y must also approach zero for the instability to occur, and this is strongly influenced by roll acceleration.

When the damping term ν_m is included in the angle-of-attack convergence solution derived in Ref. 1, this result takes the form

$$\frac{\bar{\theta}}{\theta_0} = (1 + \sigma^2)^{-1/4} \times \exp\left\{-\frac{1}{2}\int_0^t \left[\frac{\dot{p}}{p} + \nu \pm \frac{\left(\frac{\dot{p}}{p} + \nu - \frac{2}{\mu}\nu_m\right)}{(1 + \sigma^2)^{1/2}}\right] dt'\right\} \tag{7}$$

The effective damping term $\dot{p}/p + \nu$ that first appears under the integral equally damps both precessional motions. However, the relative damping of the two modes is indicated by the numerator of the expression following the $\pm$ sign, which is identically the stability criterion. When $\xi = 0$, this expression is zero and both modes are damped equally. When $\xi > 0$, the positive mode damps more rapidly, indicating a transition from positive mode to negative mode precession and when $\xi < 0$ the transition is in the other direction. Thus, Eq. (7) substantiates the conclusions derived from the phase plane analysis.

The author wishes to thank Murphy for his suggestions that have permitted the more general analysis of the problem presented above.

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Comments on "Temperature Laws for a Turbulent Boundary Layer with Injection and Heat Transfer"

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ISAACSON and AlSaji[1] have derived the expression

$$2u_\tau/Pr_t v_w\ [(1 + v_w t^+/u_\tau)^{1/2} - 1] = A\ \ln(u_\tau y/\nu) + B \tag{1}$$

for the dimensionless temperature t^+ in the inner layer of a transpired boundary layer: Pr_t is the turbulent Prandtl number, A is the reciprocal of the von Karman constant K, and B is in general a function of v_w/u_τ. The results presented by Isaacson and AlSaji, and the further results in Dr. AlSaji's thesis,[2] suggest that B is constant to within experimental error. The formula quoted above was derived from "mixing length" arguments with the further assumption that the ratio of shear stress τ to heat transfer q is independent of y. This last assumption is correct only for $Pr = 1$, $Pr_t = 1$, and it seems worth while to put on record the more accurate form of the "mixing length" temperature profile. The mixing length assumptions are discussed in Ref. 3.

In the inner layer the mean momentum and mean temperature equations reduce to

$$v_w \partial u/\partial y = \partial\tau/\partial y \tag{2}$$

$$v_w \partial T/\partial y = -\partial q/\partial y \tag{3}$$

suppressing density and specific heat, and the mixing length formulae for velocity and temperature are

$$\partial u/\partial y = \tau^{1/2}/Ky \tag{4}$$

$$\partial T/\partial y = -q/\tau^{1/2}K_\Theta y \tag{5}$$

where K/K_Θ is equal to Pr_t by definition of the latter. Isaacson and AlSaji find $K_\Theta = 0.464$; K is about 0.41 (0.424 according to AlSaji).

From Eq. (2–5)

$$\frac{\partial u/dy}{\partial T/\partial y} = \frac{-\partial\tau/\partial y}{\partial q/\partial y} = \frac{-K_\Theta}{K}\frac{\tau}{q} \tag{6}$$

$$\therefore d\tau/dq = K_\Theta \tau/Kq$$

so

$$\frac{q}{q_w} = \left[\frac{\tau}{\tau_w}\right]^{K/K\Theta} \cdot \text{const}$$

The constant, which represents the effect of the viscous sublayer in which the mixing length formulae are not valid, is unity only if $Pr = 1$, otherwise, the behavior of the heat flux and shear stress in the viscous sublayer will be different. For small v_w/u_τ, $q \to q_w$, and $\tau \to \tau_w$, so the constant is of the form

$$1 + f(v_w/u_\tau) \text{ where } f(0) = 0$$

By an analysis parallel to that of Ref. 1 we obtain

$$t^+ = \frac{u_\tau}{v_w}\left[\left(1 + \frac{u}{u_\tau}\cdot\frac{v_w}{u_\tau}\right)^{Pr_t}(1 + f) - 1\right] \tag{7}$$

so that

$$\frac{2u_\tau}{v_w}\left[\left(\frac{1 + v_w t^+/u_\tau}{1 + f}\right)^{1/2Pr_t} - 1\right] = \frac{1}{K}\ln\frac{u_\tau y}{\nu} + B' \tag{8}$$

where B' is the additive constant in the logarithmic velocity profile, which is in general a function of v_w/u_τ but which seems to be nearly constant for $v_w \geqslant 0$.[2,4–6]

This equation differs from that of Ref. 1 in having $1/Pr_t$ as a factor in the exponent on the left-hand side, and in the presence of the $(1 + f)$ factor. By requiring compatibility with

$$t^+ = (1/K_\Theta)\ln\frac{u_\tau y}{\nu} + B \tag{9}$$

as $v_w/u_\tau \to 0$ we find that f must be

$$(v_w/u_\tau)(B - B') + 0(v_w/u_\tau)^2 \tag{10}$$

near $v_w = 0$, where B' is of course evaluated at $v_w = 0$. The best estimate for $B - B'$ is about -1.1 in air (Pr = 0.7) ac-

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